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THOMAS YOUNG'S THEORY OF THE ARCH: THERMAL EFFECTS

Introduction.

Thomas Young (1773–1829), Fig. 1, was a genius, a polymath who contributed at the highest level in many fields: science, applied science, medicine, philology, Egyptology, geophysics[1]. However, he followed a heterodox way in his entire career; he was an autodidact and worked always outside the conventions of academies and universities. It may be for this reason that, freed from the discipline to explain and to teach, he was not concerned with the transmission of his discoveries. His methods were often completely original, and his way of thinking did not pertain to the usually accepted (academical) frames of thinking. The consequence was (and is) that it is extremely difficult to follow his deductions, sometimes even to know what he was looking for. Anyone who has studied Young would agree with George Peacock's description: «Important and difficult steps are passed over as manifest, terms are neglected as insignificant, analogies take

Figure 1. Portrait of Thomas Young (frontispiece of G. Peacock, *Life of Thomas Young*, 1855)

[1] The most complete and useful biographies on Young are: G. Peacock, *Life of Thomas Young*, London, John Murray, 1855, including many original documents today missing; A. Wood, *Thomas Young. Natural philosopher, 1773-1829*, completed by F. Oldham, with a memoir of A. Wood by C. E. Raven, Cambridge, Cambridge University Press, 1954; A. Robinson, *The last man who knew everythin: Thomas Young, the anonymous polymath who proved Newton wrong, explained how we see, cured the sick, and deciphered the Rosetta stone, among other feats of genius*, New York, Pi Press, 2005.

the place of proofs, and we are surprised to find ourselves at the end of an investigation, even with the limits of space which would commonly be deemed hardly sufficient to master the difficulties which meet us at the beginning»[2].

Young was also heterodox in his way of publishing; more than one half of his entire production was published anonymously. After quitting his chair in the Royal Institution and the publication of his *Lectures on Natural Philosophy* in 1807[3], all the writings not directly related with medicine or his work in the Royal Society were published anonymously, using a pseudonym or signing with letters. He feared that the diversity of his enquiries would damage his reputation and work as physician. Only when he felt that his economical position was assured, he lifted the veil of secrecy, a few years before his death in 1829. It was too late, and many of his discoveries remained during years buried in Journals and, particularly, in some of the sixty one articles he wrote for the *Encyclopaedia Britannica*. It often occurred that he received recognition in foreign countries, namely France and Germany, rather than in Great Britain. In 1855 G. Peacock published his *Life of Thomas Young*[4] and the *Miscellaneous Papers*[5], in which most of his anonymous contributions were republished. But by then, his laconic and obscure way of expressing his thoughts was even less acceptable, as the fields in which he had made pioneering contributions had developed and acquired an established frame of expression.

1. *Structural theory: The theory of the arch.*

What has been said applies particularly well to his work in the field of Structural Theory. His fundamental contribution to Strength of Materials, the definition of the modulus of elasticity (Young's modulus), obscured other fundamental discoveries in the same field. Even in this case, complete recognition came late and not until Saint-Venant published his *Historique*

[2] Peacock, *Life of Thomas Young*, p. 416.

[3] T. Young, *A course of lectures on natural philosophy and the mechanical arts*, London, Joseph Johnson, 1807. (Republished by Thoemmes Press, 2002). Young published previously an extensive syllabus: T. Young, *A syllabus of a course of lectures on natural and experimental philosophy*, London, Press of the Royal Institution, 1802. The best study of the genesis of his *Lectures* in: G. N. Cantor, *Thomas Young's lectures at the Royal Institution*, «Notes and Records of the Royal Society of London», XXV (1970), pp. 87-112.

[4] See note 1.

[5] T. Young, *Miscellaneous works of the late Thomas Young, M.D., F.R.S.*, London, John Murray, 1855. (Republished by Thoemmes Press, 2003).

abregée in 1864, did Young receive universal credit[6]. (Navier did not cite Young in his *Resumé des leçons* of 1826[7], nor most English authors after the 1830's, until Rankine's *Manual of Applied Mechanics* in 1858[8]).

But, as has been said, the definition of the modulus of elasticity was only a little part of Young's contribution to the Strength of Materials. The section «Of the equilibrium and strength of elastic substances»[9] contains important original work, among it the calculation of the buckling force of a column with imperfections. However, Todhunter in his *History of the theory of elasticity*, after revising succinctly this section, grumbled: «The whole section seems to me very obscure like most of the writings of its distinguished author; among his vast attainments in sciences and languages that of expressing himself clearly in the ordinary dialect of mathematicians was unfortunately not included»[10].

Apparently, nobody until Timoshenko[11] in 1953 bothered to understand what was contained in the cited section (a section of the Mathematical Elements of his *Lectures*; this whole part was suppressed in Kelland's edition of 1845[12]). Some forty years after, recent papers have confirmed the importance of Young's work in the instability of imperfect struts[13].

[6] B. Saint-Venant, 1864. *Historique abregé des recherches sur la résistance et sur l'élasticité des corps solides*, in E. M. Navier, *Resumé des leçons,* 3rd edition, Paris, Dunod, 1864, pp. XC-CCCXI.

[7] C. L. M. H. Navier, *Resumé des leçons données à l'École des Ponts et Chaussées, sur l'application de la mécanique à l'établissement des constructions et des machines*, Paris 1826 (2nd edition 1833; 3rd edition, with notes and appendices by B. de Saint-Venant, 1864).

[8] W. J. M. Rankine, *A Manual of applied mechanics*, London, Charles Griffin, 1858.

[9] T. Young, *Mathematical elements of Natural Philosophy. Part II. Mechanics*, in *A Course of Lectures on Natural Philosophy*, London 1807, Vol. 2, pp. 46-51.

[10] I. Todhunter – K. Pearson, *A history of the theory of elasticity and of the strength of materials from Galilei to Lord Kelvin*, New York, Dover, 1960 (first ed. 1886), Vol. 1, pp. 82–83.

[11] S. P. Timoshenko, *History of stregnth of materials. With a brief account of the history of theory of elasticity and theory of structures*, New York, McGraw-Hill, 1953, pp. 90-98.

[12] T. Young, *A course of lectures on natural philosophy and the mechanical arts. A New Edition, with references and notes by the Rev. P. Kelland*, London, Taylor and Walton, 1845. Kelland not only suppressed the *Mathematical elements of Natural Philosophy. With references to particular passages and occasional abstracts and remarks*, pp. 1-86, but, what is worse, he divided (including notes and a short bibliography at the end of each lecture) and abridged considerably the *Catalogue of Works relating to Natural Philosophy, and the Mechanical Arts*, pp. 81-520, with some 20000 entries, which constitute perhaps the most extense and useful compilation of the Literature at his time, but which includes abstracts and remarks crucial to understand Young's work. These asbtracts and remarks were eliminated completely.

[13] See: J. C. Chapman – D. Buhagiar, *Application of Young's buckling equation to design against torsional buckling, structures and buildings, Proc. of the ICE, CIC,* 1993, pp. 359-369;

It is this attitude which still pervades and which has precluded the necessary effort of studying Young's contributions. In fact, it has recently been discovered the importance of Young's contribution, not only in the field of Strength of Materials, but also in the Theory of Structures[14]. The present author has discovered that the article *Bridge*[15], published in 1817 for the *Supplement* to the Fourth edition of the Encyclopaedia Britannica, contains the first correct Theory of Arches[16] using for the first time the concept of 'line of thrust' and relating it with statements of stability and strength (Figure 2). Young's paper tackled also for the first time difficult aspects of bridge design, as the influence of a point load placed anywhere in the arch. Young considered this paper one of the main contributions he made to the Encyclopaedia and so it appears in capital letters (together with *Egypt* and *Tides*) in the list of articles which he included in his *Autobiographical sketch* discovered by Hilts in the 1970's[17].

A. N. Beal, *Who invented Young's Modulus?*, «Structural Engineer», LXXII (2000), pp. 27-32.

[14] A. N. Beal, *Thomas Young and the theory of structures 1807–2007*, «Structural Engineer», LXXV (2007), pp. 43-47. This paper presents a short review of Young's contribution based exclusively in the contents of the *Lectures*, and ignoring later contributions.

[15] T. Young, *Article 'Bridge'*, in *Supplement to the fourth, fifth and sixth editions of the Encyclopaedia Britannica*, Edinburgh, Archibald Constable, 1824 (1st printing 1817), pp. 497-520; pp. 42-44. (Republished in part in *Miscellaneous Works*, edited by G. Peacock, London, John Murray, 1855, Vol. 2, pp. 194-247.)

[16] S. Huerta, *Thomas Young's theory of the arch: His analysis of Telford's design for an iron arch of 600 feet span*, in *Essays in the history of the theory of structures, in honour of Jacques Herman*, edited by S. Huerta, Madrid, Instituto Juan de Herrera, CEHOPU, 2005, pp. 189-233. There is no reference to Young's theory of the arch in any of the previous histories of arch theory. The author was aware of Young's article as it was briefly cited in the discussion by Dorn of the debate about the feasibility of Telford's design in 1801: H. I. Dorn, *The art of building and the science of mechanics. A study of the union of theory and Practice in the early history of structural Analysis in England*, Ph.D. diss., Princeton University, 1970, pp. 190-198 (in fact, Dorn concentrated mainly on Robison's contributions and the reports of other experts of the Select Committee and did not comment Young's article). K.-E. Kurrer has acknowledged the importance of Young's contribution and has put it within the broadest frame of the History of Structural Theory: K.-E. Kurrer, *The history of the theory of structures. From arch analysis to computational mechanics*, Berlin, Ernst und Sohn, 2008, pp. 86-89. The late Sir Allan Muir Wood discussed Young's contribution and contrasted it with the Brunels practice of arch analysis: Sir A. M. Wood, *Thomas Young and the Brunels: masters of masonry analysis, Civil Engineering, Proc. of the ICE*, CLXII (2009), pp. 42-48 .

[17] V. L. Hilts, *ThomasYoung's 'autobiographical sketch'*, «Proceedings of the American Philosophical Society», CXXII (1978), pp. 248-260.

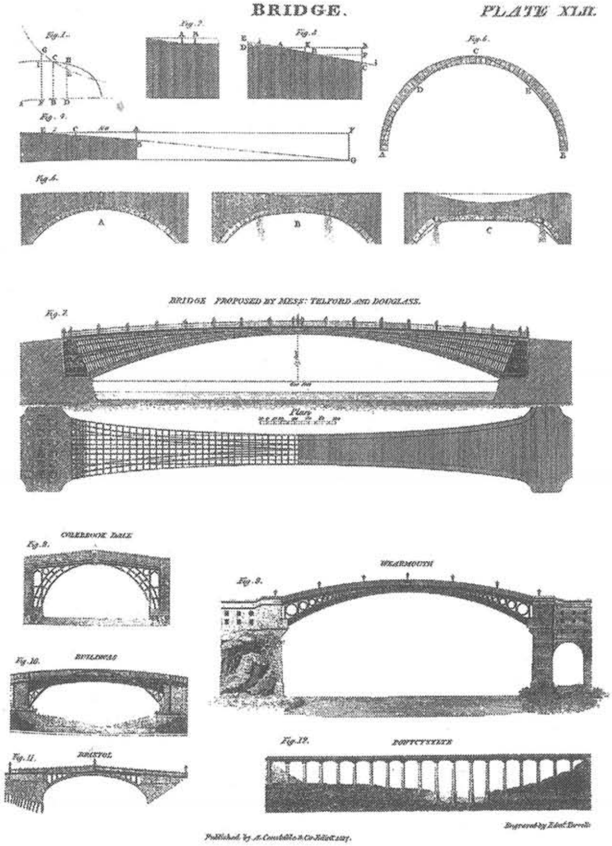


Figure 2. First Plate of the article *Bridge* by Thomas Young for the *Supplement* to the Encyclopaedia Britannica in 1817.

The matter has been discussed in full by the author in the cited paper. We will resume briefly the main aspects. His main contribution, the one which permitted him to develop and apply his theory, was to free the «curve of equilibrium» from the intrados of the arch. All the previous authors on the «theory of equilibration» in England assumed that the form of the curve of equilibrium had to coincide with the intrados. This theory followed Robert Hooke's statement of 1675: «As hangs the flexible line, so but inverted will stand the rigid arch»[18]. The

[18] An excellent historical discussion on arch theory is in: J. Heyman, *Coulomb's memoir on statics: an essay in the history of Civil Engineering,* Cambridge, Cambridge University Press, 1972, pp. 162-189. (Reprinted by The Imperial College, London, 1997).

statics of arches and hanging cables are identical. English mathematicians and engineers with a mathematical background found there a good field to test the new tools of Newtonian *calculus*, and produced a series of theoretical works without any practical use (Figure 3)[19].

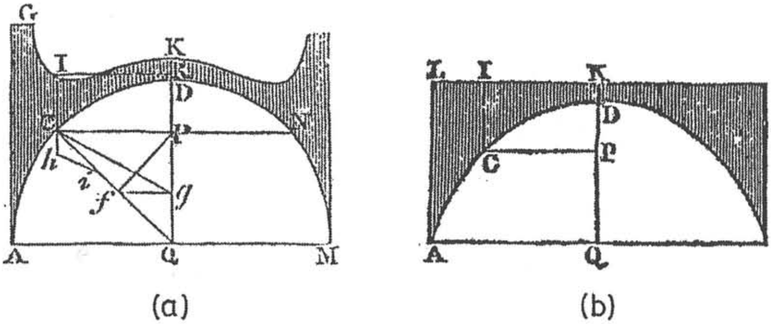


Figure 3. The two main problems of the equilibration theory: (a) to find the curve of extrados (load curve) for a given intrados; (b) to find the form of an intrados for a given extrados (Hutton, *Principles of Bridges*, 1812).

John Robison was the first, in the article *Arch* for the *Supplement* to the Third Edition of the Encyclopaedia Britannica[20], to criticize the approach - the arch will be in a state of «tottering equilibrium» - and to consider the crucial effect of the friction between the stones, but he did not grasp the true nature of the transmission of the internal forces. Robison's work inspired the first thoughts on arches by Young, gathered in his *Lectures*. But in the article *Bridge* Young expressed clearly the meaning of the concept of the line of thrust (or «curve of equilibrium», as he called it): «(...) [it] represents, for every part of a system of bodies supporting each other, the general direction of their mutual pressure»[21]. He explained that friction was crucial to free the line of thrust from the geometry of the joints (in a no-friction material, stability is only assured making the thrusts normal to the joints). Young is

[19] See for example, C. Hutton, *Tracts on mathematical and philosophical subjects comprising, among numerous important articles, the Theory of Bridges*. London, Wilkie and Robinson, 1812. For a short review of the 'theory of equilibration' see: T. Ruddock, *Arch bridges and their builders, 1735–1835*, Cambridge, Cambridge University Press, 1979, pp. 46-48; p. 204.

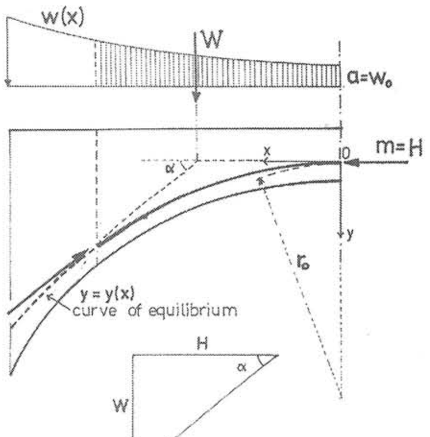
[20] J. Robison, *Article 'Arch'*, in *Supplement to the third edition of the Encyclopaedia Britannica*, Edinburgh, 1801. Republished in J. Robison, *A System of mechanical philosophy*, Edinburgh, John Murray, 1822, Vol. 1, pp. 616-60.

[21] Young, *Bridge*, p. 501.

explicit: «(...) the direction of the joints can never determine the direction of the curve of equilibrium crossing them, since the friction will always enable them to transmit the thrust in a direction varying very considerably from the perpendicular»[22].

He was well aware that the mathematical form of the curve depends on the family of planes of section considered, but he also knew that for arches the variations are very small: «(...) it is obvious that the forces represented by the various curves may vary very sensibly in their proportion, when we consider their joint operation on a vertical or on an oblique plane; although, if the depth of the substance be inconsiderable, this difference will be wholly imperceptible»[23].

He, then, considered vertical planes of section to simplify the mathematical expression of the curve and wrote the differential equation of equilibrium (see Fig. 4). After considering the usual theoretical problems (form of the curve for different forms of intrados and extrados, see Fig. 3 above), he concluded that for practical purposes, the weight of the bridge could be well represented by a parabolic load. This permits the easy integration of the differential equation of the curve to obtain a fourth degree curve. (This approach may be used today with advantage in most cases of analysis of masonry bridges, as the true arch forms a surbaissé arch



general equation of the curve of equilibrium

$$\int w dx = m\frac{dy}{dx}$$

radius of curvature of the curve of equilibrium

$$r = \frac{m}{w}\left(1+(\tan\alpha)^2\right)^{3/2} = \frac{m}{w}(\sec\alpha)^3$$

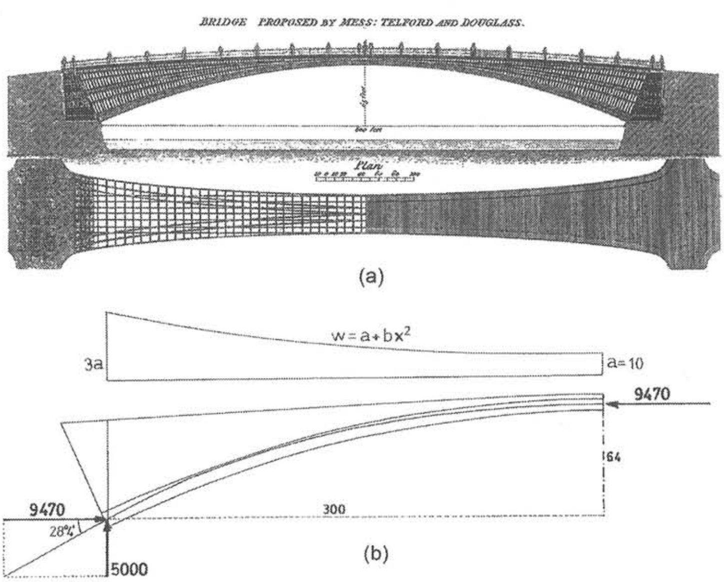
$$r_0 = \frac{m}{a}$$

Figure 4. Mathematical expression of the «curve of equilibrium» (line of thrust).

[22] *Ibidem*, p. 505.
[23] *Ibidem*, p. 501.

among the solid springings raising usually one half of the height of the arch above the imposts.)

Then, as a «tour de force», he applied his arch theory to answer the 22 questions posed in 1801 by the Select Committee to a panel of experts to evaluate the feasibility of Telford's design for an iron arch bridge of 600 feet over the Thames ('fig. 7' in Fig. 2 above)[24]. The panel, which gathered the best scientists, engineers and practitioners on bridge building and theory, was unable to give a coherent answer. Peacock resumed the failure of the



curve of equilibrium:

$$y = \frac{1}{1875}x^2\left(1 + \frac{1}{270{,}000}x^2\right)$$

(c)

Distance x.	Versed sine of the intrados.	Versed sine of the circular arc.	Ordinate y.
50	1.73	1.71	1.34
100	6.94	6.82	5.38
150	15.66	15.48	13.00
200	28.18	27.70	24.50
250	44.42	43.81	41.01
300	65.00	64.00	64.00

(d)

Figure 5. Line of thrust in Telford's iron arch design (S. Huerta, *Thomas Young's theory of the arch*, 2005): (a) Telford's design from Fig. 2; (b) drawing of the line of thrust or curve of equilibrium; (c) Young's mathematical expression of the curve; (d) ordinates of the curve calculated by Young.

[24] Huerta, *Thomas Young's theory of the arch*, pp. 209-225. The complete list of questions in *Ibidem*, pp. 227-229.

experts: The answers which were given were singularly humiliating to the pride of philosophy: «they were not only altogether at variance with each other, but in very instance incomplete and unsatisfactory»[25].

Young answered in turn every one of the questions and concluded that the design would have been safe. In particular he calculated the equation of the line of thrust for the dead load, checked that the deviation of the line would not give rise to unacceptable stress, and, finally, calculated the effect of a point load of 100 tons placed at one quarter of the span, obtaining the ordinates of the distorted line of thrust and checking again the stress level. Such a complete and correct analysis of an arch bridge was made only 50 years after, when authors like Rankine, Winkler or Castigliano published his contributions on bridge analysis.

In summary, with regard to Arch Theory, we can affirm that Thomas Young has a deep understanding of the concept of 'line of thrust'. He was

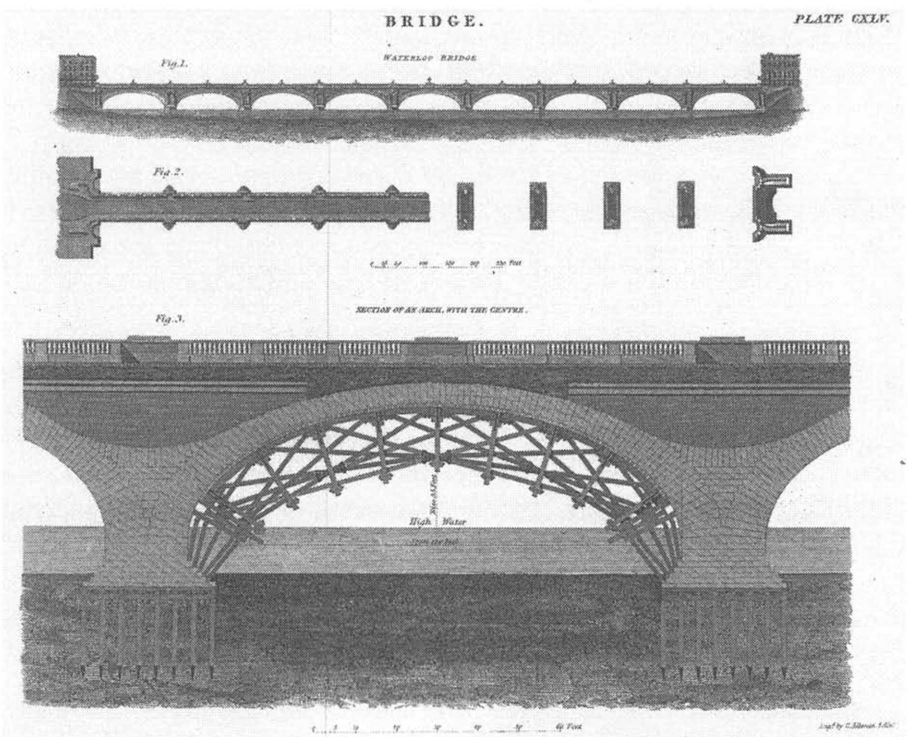


Figure 6. Line of thrust in one of the stone arches of Waterloo Bridge. The first line of thrust drawn for an arch bridge (T. Young, *Bridge*, 1824)

[25] Peacock, *Life of Thomas Young*, p. 422.

the first to free the curve of equilibrium from the «straitjacket» of the intrados (or the middle line) and employed this concept nearly twenty years before other authors. He obtained the correct general mathematical expression of the curve of equilibrium for different types of loads with a view to its application in Bridge analysis. His use of a simple parabolic load is remarkable for its simplicity and applicability in most cases. Young was also the first to study the influence on the stability of the arch of a point load placed anywhere on the extrados, and devised a completely original method of obtaining the corresponding curve of equilibrium transforming that of the dead load. Finally, he applied his theory of arches to ascertain the safety of Telford's grand design of an iron bridge arch of unprecedent size and complexity. Young's analysis is completely correct, combining statements of equilibrium (curves of equilibrium for the given loads) with statements about the material (cast iron must work in compression; therefore the curve of equilibrium must lie within the arch). The notion of a geometrical factor of safety is implicit in many of his statements. Young's theory lies completely within the modern frame of the 'limit analysis of arches' developed mainly by Heyman since the 1960's. It is not an 'elastic' analysis but an 'equilibrium analysis' validated by the 'safe theorem of limit analysis'[26].

2. *Thermal effects on arches.*

Young studied, also, the effect of thermal variations on the thrust of arches. This part is one of the most difficult of the article *Bridge* and for this reason it was not ready to be included in the cited paper on Young's theory of the arch. The author presented his research in a keynote lecture in 2006, and this constitutes the substance of what follows[27].

Architects and engineers have been always aware of the movements experimented by buildings due to changes of temperature. For example, the cracking of the dome of Saint Peter in Rome was attributed ca. 1740 by several experts to thermal effects. The different expansion coefficients of stone and iron were also (at least qualitatively) known, and when Poleni recommended in 1743 to put iron rings outside the dome he observed that they should be

[26] The importance of the Plastic theory for an understanding of structural behaviour has been stressed many times by Professor Heyman. He sees the Safe Theorem as: «The rock on which the whole theory of structural design is now seen to be based» (J. Heyman, *The science of structural engineering*, London, Imperial College Press, 1999, p. 101).

[27] S. Huerta, *The first thermal analysis of an arch bridge: Thomas Young 1817, Proceedings of the First International Conference on Advances in Bridge Engineering. Bridges – Past, Present and Future*, edited by A. Kumar *et alii*, London, Brunel University Press, 2006, vol. I, pp. 18-29.

placed in summer[28]. Movements in stone bridges were also reported and, for example, in 1824 the French engineer Vicat registered the movements of the Pont de Souillac, which he attributed to changes of temperature. He made, also, a crucial observation. The actual state of the bridge was always changing, and suggested that this should be taken into account: « (...) les grandes voûtes exposées à toutes les intempéries ne sont jamais en équilibre. Je laisse aux savans qui sont particulièrement occupés des conditions de cet équilibre, à discuter l'influence perturbatrice des mouvemens dont je viens de constater la realité»[29] (The complete answer to the question of the 'actual' state of a structure was only possible within the frame of Limit Analysis and the Safe Theorem, as professor Heyman has discussed in many of his publications[30]). In Britain, the collapse of several iron bridges ca. 1800 led to some engineers to the conclusion that the failures were due to the expansion of iron[31] and in the reports written about the feasibility of Telford's grand design for an iron arch of 600 feet over the Thames in London, one of the arguments against the proposal was the thermal effects[32]. It appears that the first systematic study of the movements of an iron bridge due to changes in temperature was made by George Rennie in 1818 during the construction of the arches of Southwark Bridge[33]. However, no structural analysis was made to explain the results. In the classical books on the history of the theory of structures by Todhunter and Pearson[34], Timoshenko[35] and Charlton[36], the first analysis of the thermal effects in an arch is attributed to Bresse[37] in 1854 and, indeed, the book by Bresse constitutes an exhaustive study of the elastic arche, though he only considered

[28] G. Poleni, *Memorie istoriche della Gran Cupola del Tempio Vaticano*, Padova, Nella Stamperia del Seminario, 1748.

[29] L. J. Vicat, *Note sur un Mouvement périodique observé aux voûtes du pont de Souillac*, «Annales de chimie et de physique», XXVII (1824), pp. 70-79.

[30] J. Heyman, *Structural analysis: a historical approach*, Cambridge, Cambridge University Press, 1998.

[31] Ruddock, *Arch bridges and their builders*, pp. 167.

[32] *Fourth Report*, *Fourth report from the select committee upon the improvement of the port of London (3rd June 1801)*, «Reports of committees of the House of Commons 1715–1801», XIV (1803), pp. 604-635, plates.

[33] G. Rennie, *On the expansion of arches*, «Transactions of the Institution of Civil Engineers», III (1842), pp. 201-218.

[34] I. Todhunter - K. Pearson, *A history of the theory of elasticity*, pp. 356-357.

[35] Timoshenko, *History of stregnth of materials*, p. 149.

[36] T. M. Charlton, *A history of the theory of structures in the nineteenth century*, Cambridge, Cambridge University Press, 1982, p. 42.

[37] J. A. C. Bresse, *Recherches analytiques sur la flexion et la résistance des pièces courbes, accompagnées de tables numeriques pour calculer la poussée des arcs chargés de poids d'une*

thermal effects in two-hinged arches. Rankine, in his *Manual of civil engineering* of 1862, made an elastic analysis of arches, including for the first time encastré arches, but did not considered thermal effects[38]. Winkler considered the effects of changes of temperature in his manual of 1869 for all types of end conditions[39]. In the last quarter of the 19th century the study of the thermal effects in elastic arches appears in many texts on structural analysis. The book by Castigliano[40], published in 1879, is particularly useful for the clarity of exposition and the calculation examples.

However, Thomas Young (1773–1829) in the article *Bridge* written in 1817 for the *Supplement* to the 4th edition of the Encyclopaedia Britannica[41] included a correct analysis of the thermal effects on a shallow segmental arch with *encastré* ends, preceding in 50 years Winkler's analysis. In what follows a detailed study will be made of Young's pioneering contribution on the thermal analysis of arches. The part of the *Bridge* article concerning the strength of materials will be also discussed, as it is fundamental to understand Young's elastic analysis of the effect of changes of temperature.

2.1. *«Resistance of materials»: the calculation of strains.*

The article *Bridge* is divided in six parts. We are concerned here mainly with the first part, «Resistance of materials». In it Young extended his theory of «passive strength» (elasticity) already expounded in his *Lectures* of 1807 with a view to its application to arches[42]. Young makes an effort to explain the theory in rigorous terms. The method used by Young is the 'classical' method of stating a proposition (named alphabetically from A to Z) and then

manière quelconque et leur pression maximum sous une charge uniformément répartie, Paris, Mallet-Bachelier, 1854.

[38] W. J. M. Rankine, *A manual of Civil Engineering*, London, Griffin Bohn and Company, 1863[2] (first ed. 1862), pp. 296-314.

[39] E. Winkler, *Die Lehre von der Elasticitaet und Festigkeit, mit besonderer Rücksicht auf ihre Anwendung in der Technik*, Prag, Dominicus, 1867, pp. 358-369.

[40] C. A. P. Castigliano, *Théorie de l'equilibre des systémes élastiques et ses applications.* Turin, Augusto Federico Negro, 1879 (Translated by E. S. Andrews in *Elastic stresses in structures*, London, Scott, Greenwood and Son, 1919. Republished with an introduction by G. A. Oravas in *The theory of equilibrium of elastic systems and its applications*, New York, Dover, 1960).

[41] T. Young, *Bridge*, in *Supplement to the fourth, fifth and sixth editions of the Encyclopaedia Britannica,* Edinburgh, Archibald Constable, 1824 (1st printing 1817, republished in part in: *Miscellaneous Works,* edited by G. Peacock, London, John Murray, 1855, 2, p. 194-247), Vol. 2, p. 497-520; p. 42-44.

[42] T. Young, *A course of lectures on Natural Philosophy and the Mechanical Arts*, London, Joseph Johnson, 1807, (Reprint: Bristol, Thoemmes Press, 2002).

demonstrating it. This way of exposition makes difficult to follow the general line of reasoning and results particularly exasperating to a modern reader. The propositions though formulated in a general manner are directed to study the arch problem. In this part Young is treating two problems: a) the eccentric compression of a block and the calculation of the resulting stresses; b) the stresses due to thermal effects. We will follow Young's order of exposition.

First he states the proportionality between tensions and deformations and to justify this he expounds a theory of cohesive and repulsive molecular forces and states that even if the law of this forces is not linear ('fig. 1' in Fig. 7), the effect will be proportional for a small «change of dimensions».

He then treats the eccentrical compression of a block and begins considering the limit position of an eccentric force so that all the section remains in compression and the corresponding increase in the stresses. However, the way he expressed the problem is as follows: «The strength of block or beam must be reduced to one half, before its cohesive and repulsive forces can both be called into action»[43].

A modern engineer may have no difficulty in interpreting this: Young is obviously referring to the 'middle third' concept and the maximum stress is double as the mean stress. To demonstrate this Young assumes explicitly that plane sections remain plane after the deformation. It follows that the deformations vary linearly: « (...) and consequently the forces may always be represented, like the pressure of a fluid, at different depths, by the ordinates of a triangle; and their result may be considered as concentrated in the centre of gravity of the triangle, or of such of its portions as are contained within the depth of the substance»[44].

Here Young is struggling with the concept of stress and he uses the analogy of the pressure of a fluid. However he tries always to speak in terms of deformations, the 'forces' or 'pressures' being always proportional to them, as stated in the first proposition ('fig. 2' in Fig. 7).

The next proposition states that: «The compression or the extension of the axis of the block or beam is always proportional to the force, reduced to the direction of the axis, at whatever distance it may be applied»[45]. The deformation of the axis is always equal to the mean deformation, produced by the normal component of the force applied in the middle of the section. The transverse component of the force will be resisted by 'lateral adhesion' (shear) and if the force is normal to the axis, the length of the axis will remain unaltered.

[43] Young, *Bridge*, p. 497.
[44] *Ibidem*, p. 498.
[45] *Ibidem*.

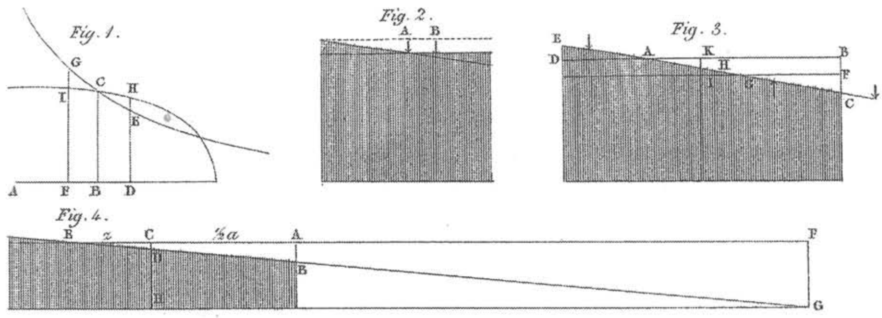


Figure 7. Drawings on *Resistance of materials*. Young is concerned with the 'compression' or 'extension' of a joint which remains plane after deformation (Young, *Bridge*, 1824).

Then Young proceeds to locate the neutral point for this general force placed at any distance: «The distance of the neutral point from the axis is to the depth, as the depth to twelve times the distance of the force, measured in the transverse section»[46]. In an algebraical form:

$$z = \frac{a^2}{12y} \tag{1}$$

where z is the distance of the neutral point from the axis, a is the depth of the section and y is the distance of the point of application of the force to

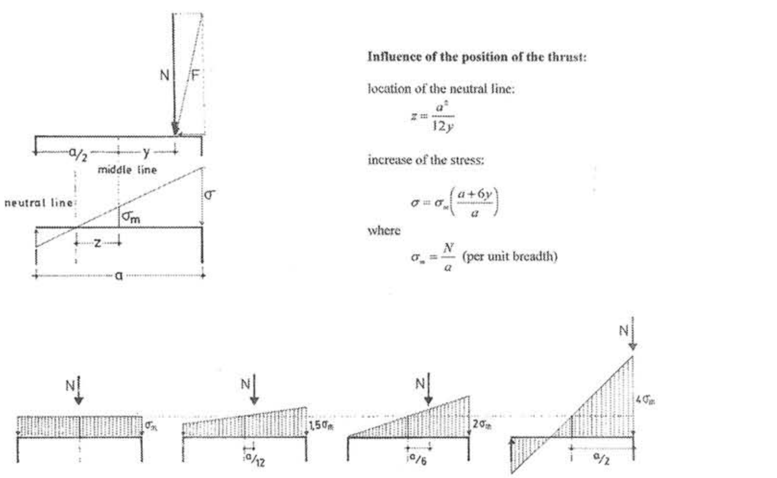


Figure 8. Young's propositions on *Resistance of materials* expressed in modern terms of stresses (Huerta, *Thomas Young's theory of the arch*, 2005).

[46] *Ibidem.*

the axis (Fig. 8). Young's demonstration is based in the proportionality of the stress resultants and the triangular form of the stress blocks.

The next proposition tries to relate the increase of the normal stresses in terms of the distance of the force from the axis (Fig. 8): «The power of a given force to crush a block, is increased by its removal from the axis, supposing its direction unaltered, in the same proportion as the depth of the block is increased by the addition of six times the distance of the point of application of the force, measured in the transverse section»[47].

Young is referring to the increase of the strains (stresses) due to the eccentricity of the load. In modern terms, if we call σ_m the mean compressive stress produced by the force applied in the centre of the section, the removal of the force at a distance y will produce a stress σ given by:

$$\sigma = \sigma_m \left(\frac{a + 6y}{a} \right) \tag{2}$$

Young demonstrates the assertion, again, for similar triangles ('fig. 3' in Fig. 7). Therefore, now we are in the situation to ascertain the 'strength' of any section acted by any force located at any distance, comparing the maximum deformation (strain) of the section with the corresponding fracture value for the material. This was the objective of the first four propositions.

Now Young turns to the study of the flexural deformation of the blocks or sections of the arch. He begins studying the curvature produced in the neutral line by any given force: «The curvature of the neutral line of a beam at any point, produced by a given force, is proportional to the distance of the line of the direction of the force from the given point of the axis, whatever that direction may be»[48].

He is stating that the curvature is proportional to the bending moment produced by the force. He has already shown that the distance z of the neutral point to the axis is inversely proportional to the distance of the force y, (eq. (1)). Now, with reference to 'fig. 4' in Fig. 7, is evident that: $z/\mathrm{CD} = r/(\mathrm{CH})$ or the curvature $k = 1/r = (1/z)(\mathrm{CH}/\mathrm{CD}) = y\ (\mathrm{CH}/\mathrm{CD})$. As $(\mathrm{CH}/\mathrm{CD})$ is proportional to the force f, it follows that:

$$k = \text{Constant } (f\,y) \tag{3}$$

i.e. is proportional to the bending moment. Young do not use the term 'bending moment', but remarks that if the force f is inclined the distance y

[47] *Ibidem.*

[48] *Ibidem*, p. 499.

should be measured through the perpendicular to the direction of the force. Finally, Young states that «the radius of curvature of the axis will always be to that of the neutral line as the acquired to the original length of the axis», as is evident with reference to the cited figure.

Then, Young defines the constant which relates the curvature k with the bending moment $(f\,y)$: «The radius of curvature of the neutral line is to the distance of the neutral point as the original length of the axis to the alteration of that length; or as a given certain quantity to the external force: and this quantity has been termed the Modulus of elasticity»[49].

For this he makes use of the concept of 'modulus of elasticity', which, in modern terms relates the stresses and strains. He considers that the reader is already familiar with the concept. In fact, Young's definition of the modulus of elasticity, given in his *Lectures* is anything but clear: «The modulus of elasticity of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight of causing a certain degree of compression, as the length of this substance is to the diminution of its length»[50].

Young sometimes speaks of the 'weight of the modulus' (which is EA, being E the modern definition of the modulus and A the area of the cross section) or of the 'height of the modulus' (which is E/γ, being γ the specific weight of the material; this last definition is independent of the cross section).

Using the same equation as in the preceding proposition: $r/z = M/f$, being M $(=EA)$ the weight of the modulus. Now, as $z = a^2/(12y)$, then $r = Mz/f = M\,a^2/(12fy)$. In modern terms:

$$(fy) = (1/r)(Ma^2/12) = k\,(EI), \tag{4}$$

being I the second moment of area of a rectangle of height a and breadth unity.

The next proposition establish the relationship between the strain of the axis and the strain in the line of the direction of the force: «The flexibility, referred to the direction of the force, is expressed by unity, increased twelve times the square of the distance, divided by that of the depth»[51]. Now Young is looking after the relationship between the strain of the middle axis ε_m and the strain in the line of action of the force ε. It is evident in 'fig. 4' of Fig. 7, that $CD/z = FG/(z + y)$, then $(FG/CD) = (z + y)/z = 1 + (y/z)$, that is

[49] *Ibidem*.

[50] Young, *Lectures*, Vol. II, p. 46.

[51] Young, *Bridge*, p. 499.

$$\frac{\varepsilon}{\varepsilon_m} = 1 + 12\frac{y^2}{a^2} \tag{5}$$

If the direction of the force is oblique the relationship remains the same if we consider the strain of the axis in a direction parallel to that of the force.

2.2. *Elastic analysis of an* encastré *arch: the study of thermal effects.*

Now Young has all what he need to attack the problem of the effect of a temperature change. An increment of temperature θ will produce a strain ($\theta\ v$) where v is the coefficient of thermal expansion. In an arch of fixed abutments this strain will induce a set of internal forces which will be superposed to the internal forces due to the load. The calculation of the strain of a straight bar is easy: Young set himself a difficult problem to calculate the relationship between the strain of the cord and the maximum strain in a shallow circular arch with encastré ends. He resumed the result of his investigations in the following proposition:

If a solid bar have its axis curved a little into a circular form, and an external force be then applied in the direction of the chord, while the extremities retain their angular position, the greatest compression or extension of the substance will ultimately be to the mean compression or extension which takes place in the direction of the cord as

$$\left(1+\frac{4h}{a}\right) \quad \text{to} \quad \left(1+\frac{16h^2}{15a^2}\right)$$

a being the depth of the bar, and h the actual versed sine, or the height of the arch[52].

The solution of the problem implies the elastic analysis of a segmental circular arch with fixed ends. Young applied the 'principle of superposition' and ideas of the compatibility of deformation:

We must here separate the actions of the forces retaining the ends of the bar into two parts, the one simply urging the bar in the direction of the chord, and the other, which is of a more complicated nature, keeping the angular direction unaltered; and we must first calculate the variation of the angular situation of the ends, in consequence of the bending of the bar by the first portion, and then the strain required to obviate that change, by means of a force acting in the direction of the middle of the bar, while the ends are supposed to be fixed.

[52] *Ibidem*, p. 499.

There is no drawing in the article to explain the reader this application of the principle of superposition. The approach has been summarized in Figure 9. Young knows that the deformation must be symmetrical. First, then, he calculates the angular variation of the extreme of a semi-arch fixed at the keystone acted by a horizontal force p at the lower end, Fig. 9 (a). Calling x the angle which determines the position of a section, r the radius of curvature of the middle line of the arch and y the vertical distance of the section to the chord of the arc, then the bending moment will be p ($r\cos x - b$), where b is the cosine of the whole semi arc c ($b = r\cos c$). If we call i the inclination of the bar, the curvature $k = di/ds = p\ (r \cos x - b)/EI$; but $ds = rdx$, so that $di = (r/EI)\ p(r \cos x - b)dx$. The angular variation will be the integral for the whole semi arc, that is, it will be proportional to

$p\ (r \sin c - bc)$,

and this is the result which Young correctly gives (he drops the constant 1/ EI).

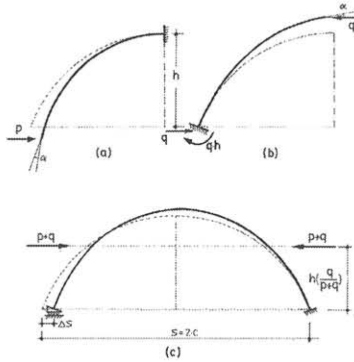


Figure 9. Study of the stresses in an arch *encastré* in both abutments due to a horizontal displacement of the abutments, maintaining the angular inclination constant.

Then he calculates the angular variation of the extreme of the semi arch fixed at the lower end due to the action of another horizontal force q acting at the superior end, Fig. 9 (b). Now the bending moment is $q(r - r\cos x)$ and, as before, $di = (1/EI)\ q(r - r \cos x)dx$. The angular variation for the whole arc, after the integration, will be proportional to $q(rc - r\sin c)$, the constant the same as before $1/EI$. For compatibility of deformation the two angular variations must be equal, therefore:

$$\frac{p}{q} = \frac{rc - r\sin c}{r\sin c - bc} \qquad (6)$$

The deduction is completely general for any arch with an opening angle $2c$. Now Young introduces the simplification for a shallow arch. When the arc is small:

$$\sin c = c - \frac{1}{6}c^3 + \frac{1}{120}c^5 \ldots$$

considering only the first two terms and supposing that $(r - b)$, the height of the arch, is very nearly $(1/2)rc^2$ (i.e. approximating the arc to a parabola), and now c representing the length of the semi chord. Substituting these values, we obtain $p/q = 1/2$.

Now, in the whole arch the two actions must be summed. At the lower ends of the arch act a force $(p + q)$ and a moment qh. We can reduce this to a single force $(p + q)$ acting at a distance $(p + q)/q = (1 + p/q)$ from the chord, Fig. 3 (c). For a shallow arch ($p/q = ½$) this distance becomes 2/3 of the height of the arch or, as Young expresses it « (...) when the force is considered as single, the distance d of the line of its direction from the summit must ultimately be one-third of the versed sine or height»[53].

Now Young turns to calculate the reduction of length of the arch chord. The differential variation of the chord of the arch due to the action of a certain force f, for a cross section A and a modulus of elasticity E, is

$$d\delta = \varepsilon_m \left(1 + 12\frac{y^2}{a^2}\right)dx = \left[\frac{f}{EA}\right]\left(1 + 12\frac{y^2}{a^2}\right)dx \qquad (7)$$

where y is the vertical distance of the middle line to the chord, i. e. the line of action of the force f.

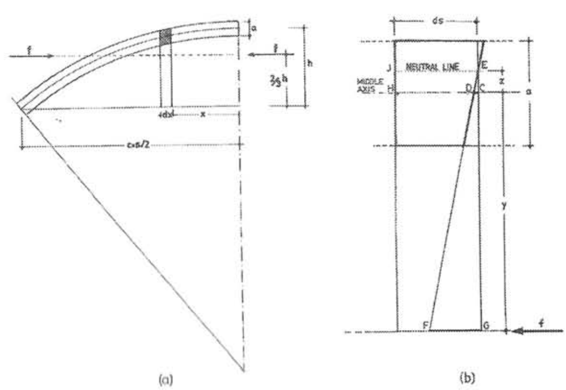


Figure 10. Deformation of an arch due to the normal strains.

[53] *Ibidem*, p. 500.

The variation of length will be the integral through the whole span $s = 2c$. As the arch is shallow, Young approximates the circular arc to a parabola, $y = (h/3)-(x^2/(2r))$, where h is the height of the arc and r is the radius of curvature at the crown. Another property of the parabola is that $h=c2/(2r)$, where c is half the chord. Substituting these values in eq. (7), and integrating, we obtain:

$$\delta = \Delta s = 2\left[\frac{f}{EA}\right]\left(c + \frac{16h^2c}{15a^2}\right) = s\left[\frac{f}{EA}\right]\left(1 + \frac{16h^2}{15a^2}\right) \qquad (8)$$

and the right bracket expresses the relationship between the variation of length of the chord of the arch and that of the straight bar of the length of the chord. For a given deformation the force f at the abutments of the arch will be reduced by this factor in relation with the force applied to the straight bar for the same deformation, and the mean stress at the crown will be reduced by the same factor.

Now this force f is located at $(2/3)h$, and the maximum stress will be at the springings and the relationship to the mean stress at the crown will be given by eq. (2). Therefore the stress at the springings will be $(1+4(h/a))$ times the mean stress (disregarding the effect of the inclination of the section, which is very small for shallow arches). Therefore the maximum stress or strain at the springing of the arch, due to a certain increment or decrement of the span of the arch, maintaining the inclination of the springings, will be

$$\sigma = \sigma_m \left[\frac{1 + \frac{4h}{a}}{1 + \frac{16h^2}{15a^2}}\right] \qquad (9)$$

2. 3. *Example of application.*

Immediately after the preceding demonstration, Young gives an example of the application of his proposition to the calculation of the maximum stress due to a certain increase (or decrease) of temperature. He considers an arch with a constant depth $a = 10$ ft. and a height $h = 20$ ft. (Figure 11); as we have seen the span s does not enter in the calculations. Then the term in brackets, which gives the relationship between the maximum stress in the springing and the mean stress at the crown, will have a value of (9/5.267) or nearly 17/10.

If the arch suffers a change of temperature of θ=32° Fahrenheit (17.8° C), this will lead to a variation of length (a 'strain' ε in modern terms) of θv = 1/5000. (In his *Lectures* he gives for cast iron $v = 6.18\ 10^{-6}$ °F, referring to the experiences of Lavoisier; so it appears he is rounding the value of $32\times 6.18\ 10^{-6} = 1/5056$.) The whole arch will try to expand or contract 1/5000, but the

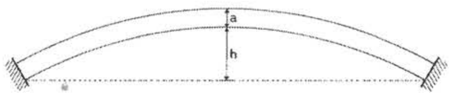


Figure 11. Surbaissé example arch

length of the chord, between the abutments must remain the same, and therefore an external force must act which will reduce its length by this amount. The maximum strain at the abutments will be 10/17 of this quantity, that is nearly 1/3000, «which is the equivalent of the pressure of a column of the metal of about 3300 feet in height, since *M*, the height of the modulus of elasticity is found, for iron and steel, to be about 10,000,000 feet»[54].

Young obtains the stress multiplying the strain by the modulus of elasticity, but he is obtaining the 'stress' not as force divided by area, but as the stress at the base of a column of constant section of certain height. It is not difficult to see that both methods lead to the same result. As we have seen, $M = E/\gamma$, then $\varepsilon M = \varepsilon\, E/\gamma = \sigma/\gamma$ which is the height of a column of uniform section build of a material of specific weight γ which presents a stress σ at its base. (This form of measuring the stresses in buildings was first proposed by Gautier in 1716 and was employed by Perronet to compare the stresses in different buildings. The breaking strength, then, was the limit height which can be build with a certain material and this parameter was still present in many engineering handbooks of the 19th century[55]).

A height of 3300 feet leads to a stress of 5 tons/sqin. or 77 N/mm^2. Young finds this stress quite high and affirms that in this case: «[it] would certainly require particular precaution, to prevent the destruction of the stones forming the abutment by a force so much greater than they are capable of withstanding without assistance».

However, immediately after, Thomas Young give us a proof of his deep insight in the behaviour of arches:

Should such a case indeed actually occur, it is probable that the extremities would give way a little, and that the principal pressure would necessarily be supported nearer the middle, so that there would be a waste of materials in a situation where they could co-operate but imperfectly in resisting the thrust ; an inconvenience which would not occur if the bar were made wider and less deep, especially towards the abutments.

[54] *Ibidem*, p. 500.

[55] S. Huerta, *Arcos, bóvedas y cúpulas. Geometría y equilibrio en el cálculo tradicional de estructuras de fábrica*. Madrid, Instituto Juan de Herrera, 2004, p. 316; pp. 362-363.

In Figure 12 we have tried to interpret Young's suggestion.

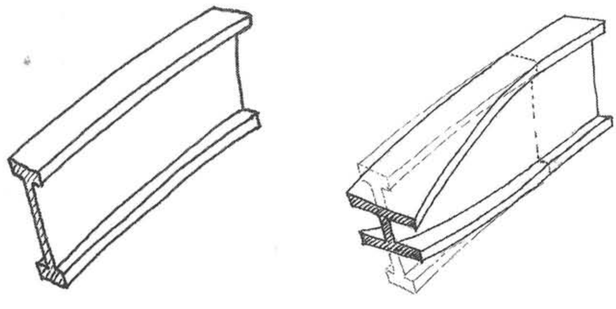


Figure 12. Hypothetical reconstruction of Young's suggestion for iron rib design at the abutments

2. 4. *A case study: Southwark Bridge.*

The last part of the article *Bridge* is dedicated to the «description of some of the most remarkable bridges which have been erected in modern times». It begins with a brief, but excellent, history of cast iron bridges, but most of the space is dedicated to a detailed analysis, employing his theory of arches and passive strength, of two bridges, the Southwark (Fig. 13) and Waterloo bridges (see Fig. 6, above). The first is of cast iron and the second of stone, and it is evident that Young wants to show that his theory can be applied to either material.

In the case of Southwark Bridge a thermal analysis is made, following his method, explained above. The span of the central arch was 240 feet and the engineer John Rennie was worried about the possible effects of thermal expansion. The Wearmouth Bridge, of similar dimensions, has presented many problems and some other cast iron bridges have collapsed. In his autobiography, he says, referring to the arches of Southwark Bridge:

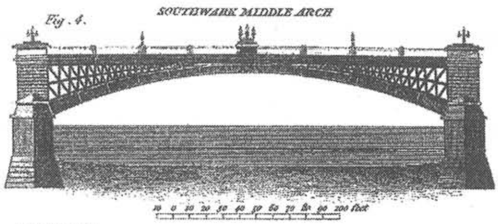
> As those arches were the largest of the kind ever constructed, considerable doubts as to their stability occurred to many, and the subject was discussed amongst scientific men with considerable energy; and amongst others, the celebrated Dr. Young undertook to investigate Mr. Rennie's calculations, and came to the conclusion that the bridge was well designed, and would be a perfectly safe structure[56].

[56] J. Rennie, *Autobiography of Sir John Rennie.* London, Spon, 1875, p. 8. See also: Ruddock, *Arch Bridges*, pp. 167-168.

It could be that Rennie is referring to the article of the Encyclopaedia Britannica, as Napier, the editor, wrote to Young just after publication: «I have been amused with two or three conjectures as to the write of this remarkable article. Rennie has been sadly puzzled by the signature O.R. which, when separated, stands for two other contributors in the list, but whose combined strength, as he rightly says, could not have produced Bridge»[57]. It is surprising that George Rennie (father of John Rennie) does not even mention Young's contribution in his paper on the expansion of arches[58]. No doubt he was unable to understand it.

Weight of half of the middle arch of Southwark Bridge.

No.	8 Blocks.		8 Oblique Stays.		Cross Frames.		Crosses.		Spandrels.		Total.	
	t.	cwt.	t.	cwt.	t.	cwt.	t.	cwt.	t.	cwt.	t.	cwt.
1	62	18	2	11	11	0	9	1	26	4	111	17
2	60	19	2	12	10	13	8	15	20	3	103	4
3	54	15	2	13	10	2	8	2	32	16	108	10
4	51	3	2	11	9	17			23	14	87	6
5	50	17	2	13	9	16			32	14	95	19
6	51	2	2	13	9	15			24	15	88	6
half 7	25	12	2	12					20	7	48	12
Covering-plates											152	0
Cornice and palisades											77	5
Roadway and pavement											650	0
Whole weight											1,523	0
Springing plate											13	10
Abutment											11,000	0



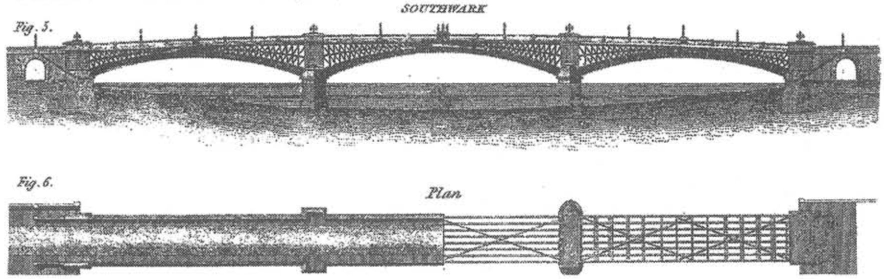


Figure 13. Southwark Bridge. Elevations and Young's estimation of the loads (Young, *Bridge*, 1824)

Southwark arches have varying section, so Young took a mean depth of $a = 7$ feet. The height of the arch $h = 23$ feet. Then, $1+ 4(h/a) = 14.14$ and $1+ (16h^2/15a^2)=12.52$, and the relationship between the maximum and mean stress is 1.129. «If, in a long and sever frost, the temperature varied from 52° to 20°, since the general dimensions will contract 1/5000, the extreme parts of the blocks near the abutments would vary 1.129/5000 of their length». Now, as the height of the modulus is 10.000.000 feet, the stress is

[57] A. Wood – F. Oldham. *Thomas Young. Natural philosopher, 1773-1829*, Cambridge, Cambridge University Press, 1954.

[58] Rennie, *On the expansion of arches*. See note 32 above.

represented by a column of 2258 feet height, that is 3 tons/sqin or 46.5 N/mm^2. Then he compares this value with the stress due to the dead load of the bridge, which he has estimated following his theory of arches in 2 tons/sqin or 31 N/mm^2. The change of temperature will more than double this stress. The concern is that such a concentration of stresses may crack the supporting stones in the abutments and Young praises Rennie's solution of «causing the blocks to bear somewhat more strongly on the abutments at the middle than at the sides, so as to allow some little latitude of elevation and depression, in the nature of the joint». I have not been able to see the actual details, but it appears that Young is referring to making the supporting joint with a little concavity, a disposition much used in the second half of the 19th century[59].

Conclusions.

Thomas Young enunciated the first correct and comprehensive arch theory, going into the detail of every aspect relevant to arch bridge design and analysis. His contribution published anonymously in the Encyclopaedia Britannica in 1817, did not receive the attention of his contemporaries. In fact, Young's contribution has been ignored in all the histories of arch theory until its rediscovery by the present author in 2005.

In his attempt to evaluate the effects of changes of temperature in iron arches, Young was the first to make an elastic analysis of an arch rib, combining statements about the material and compatibility with those of equilibrium. In this he preceded Navier by 10 years.

He made a completely correct thermal analysis of a shallow *encastré* arch and devised a simple method for calculating the increase of stresses or strains which can be used even today with advantage. His analysis preceded in some 50 years the first correct analysis of Winkler in 1867.

Though being a reputed Scientist, Young's approach to arch analysis is that of an engineer, making the necessary simplifying assumptions to obtain a safe solution for the problem in question. Young contributions to Science have been long recognised. He deserves, also, a place of honour in the History of Engineering Science.

[59] W. Lorenz, *Die Entwicklung der Dreigelenksystems im 19. Jahrhundert*, «Stahlbau», 59 (1990), pp. 1-10.

TEMI E TESTI

82

"BETWEEN MECHANICS AND ARCHITECTURE"
SERIE DIRETTA DA ANTONIO BECCHI E FEDERICO FOCE

MECHANICS AND ARCHITECTURE BETWEEN *EPISTÉME* AND *TÉCHNE*

IN COMMEMORATION OF EDOARDO BENVENUTO (1940-1998)
ON THE TENTH ANNIVERSARY
(ROME, DECEMBER 5, 2008)

edited by
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